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Original Paper

Psychophysical Threshold Estimates in Logistic Regression Using the Bootstrap Resampling

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Abstract

We propose the non-parametric bootstrap resampling algorithm for the problem of psychophysical threshold estimates. We use the logistic regression with guessing rate and the log-likelihood ratio test statistics of two samples for testing the hypothesis by using the bootstrap resampling. We apply our algorithm to the visual acuity test, and show that the bootstrap resampling is useful for the problem of the two-sample test when the numbers of observations are not identical between the two samples. We also propose the bootstrap algorithm for one-sample testing to certify the values of parameters and threshold obtained by logistic regression.

Introduction

The bootstrap resampling method provides a powerful procedure for estimating the variance of a parameter of a function. For this computer-based method we can refer to Efron et al. [1], Davison et al. [2], Foster et al. [3, 4], Joy et al. [5] and Good [6].

For the psychophysical experiment by constant stimuli method, Nagai et al. [7] proposed the statistical significance testing of difference between multiple thresholds. Bach [8], Beck et al. [9] and Schulze-Bonse et al. [10] developed the automated procedures on the personal computer for the measurements of visual acuity.

Mita et al. [11] developed a statistical method for evaluating the logarithmic visual acuity (LogVA) changes in an individual, and calculated LogVA ± *SD* (*SD* : standard deviation) by logistic regression, and also evaluated it using Nagai's test of significant difference.

The categorial data analysis and the logistic regression have been studied by McCullagh et al. [12], Christensen [13], Harrell [14] and Agresti [15].

In the present paper we propose the non-parametric bootstrap resampling for the problem of psychophysical threshold estimates. We propose the logistic regression with guessing rate and formulation of deviance residuals in sections 2 and 3. We show the log-likelihood ratio test statistics in section 4, and

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the non-parametric bootstrap resampling and testing of hypothesis in sections 5 and 6. Finally, in section 7 we present an application of our algorithm to psychophysical threshold estimates in the visual acuity test.

Logistic regression with guessing rate

We assume that the logit function is expressed in the form:

$$\mathrm{logit}\,(p_0) \equiv \log\frac{p_0}{1-p_0} = \alpha+\beta x,$$

where p_0 is the (primitive) probability, x is the explanatory variable and α, β are constants. Then p_0 is given by

$$p_0(x;\ \alpha,\ \beta) = (1+\exp(-\alpha-\beta x))^{-1}.$$

We introduce the third parameter γ ($0 \leq \gamma < 1$) for including the guessing rate. Then we have the probability p such that

$$p(x;\ \alpha,\ \beta,\ \gamma) = p_0(x;\ \alpha,\ \beta)+\gamma(1-p_0(x;\ \alpha,\ \beta)).$$

Let

$$X = \{x_j,\ \mu_j(j=1,\ 2,\ \cdots,\ N)\}$$

be the set of binomial observations where x_j (j = 1, 2, $\cdots$, N) are the explanatory variables for j-th (j = 1, 2, $\cdots$, N) observations respectively and μ_j (j = 1, 2, $\cdots$, N) are outcome data:

$$\mu_j = \begin{cases} 1 & \text{if } j\text{-th outcome is "success"}, \\ 0 & \text{if } j\text{-th outcome is "failure"}. \end{cases}$$

Then the logarithmic binomial likelihood L (α, β, γ) is given by

$$\begin{aligned} L(\alpha,\ \beta,\ \gamma) &= \log\prod_{j=1}^{N} p_j^{\mu_j}(1-p_j)^{1-\mu_j} \\ &= \sum_{j=1}^{N}(\mu_j \log p_j+(1-\mu_j)\log(1-p_j)), \end{aligned}$$

where $p_j = p$ (x_j ; α, β, γ) (j = 1, 2, $\cdots$, N). We assume that γ is a known constant γ_0 ($0 \leq \gamma_0 < 1$). Then the partial derivatives of L (α, β, γ_0) with respect to α and β are given by

$$\begin{aligned} \frac{\partial L}{\partial \alpha} &= \frac{1}{1-\gamma_0}\sum_{j=1}^{N}(\mu_j-p_j)\frac{p_j-\gamma_0}{p_j}, \\ \frac{\partial L}{\partial \beta} &= \frac{1}{1-\gamma_0}\sum_{j=1}^{N}(\mu_j-p_j)\frac{p_j-\gamma_0}{p_j}x_j, \\ E\left[\frac{\partial^2 L}{\partial \alpha^2}\right] &= -\sum_{j=1}^{N}\omega_j, \\ E\left[\frac{\partial^2 L}{\partial \alpha\,\partial \beta}\right] = E\left[\frac{\partial^2 L}{\partial \beta\,\partial \alpha}\right] &= -\sum_{j=1}^{N}\omega_j x_j, \\ E\left[\frac{\partial^2 L}{\partial \beta^2}\right] &= -\sum_{j=1}^{N}\omega_j x_j^2, \end{aligned}$$

where E [X] is the expected value of X, and ω_j is defined by

$$\omega_j \equiv \frac{1-p_j}{p_j}\left(\frac{p_j-\gamma_0}{1-\gamma_0}\right)^2.$$

We define the following notations for easy description:

$$f \equiv \frac{\partial L}{\partial \alpha}, \quad g \equiv \frac{\partial L}{\partial \beta}.$$

We shall obtain α and β by adopting the Fisher score method. Let α^t, β^t, f^t, g^t (t = 0, 1, 2, $\cdots$) be the values of α, β, f, g at iterative step t (t = 0, 1, 2, $\cdots$) and let $\alpha^0 = \beta^0 = 0$. Then we can write the algorithm for determining α and β such that

$$v^{t+1} = v^t + (F^t)^{-1} s^t (t = 0,\ 1,\ 2,\ \cdots);$$

$$v^t \equiv \begin{pmatrix} \alpha^t \\ \beta^t \end{pmatrix},$$

$$s^t \equiv \begin{pmatrix} f^t \\ g^t \end{pmatrix},$$

$$F^t \equiv -E\left[\left(\frac{\partial\,(f^t,\ g^t)}{\partial\,(\alpha^t,\ \beta^t)}\right)\right],$$

where (∂ (,) / ∂ (,)) is a Jacobian matrix. We stop the above iterative procedure if

$$\text{Norm} \equiv (v^{t+1} - v^t)^T (v^{t+1} - v^t) < \varepsilon$$

is satisfied for sufficiently small positive number ε.

Let $\hat{\alpha}$, $\hat{\beta}$, and $\hat{F}$ be the optimal values of α, β and F respectively. Then by the Cramér-Rao lower bound, we can obtain variances such that

$$(\hat{F})^{-1} = \begin{pmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha},\ \hat{\beta}) \\ \text{cov}(\hat{\beta},\ \hat{\alpha}) & \text{var}(\hat{\beta}) \end{pmatrix}$$

$$= \begin{pmatrix} (\text{se}(\hat{\alpha}))^2 & r_{\alpha\beta}\,\text{se}(\hat{\alpha})\text{se}(\hat{\beta}) \\ r_{\beta\alpha}\,\text{se}(\hat{\beta})\text{se}(\hat{\alpha}) & (\text{se}(\hat{\beta}))^2 \end{pmatrix},$$

where var $(\hat{\alpha})$ and var $(\hat{\beta})$ are variances of $\hat{\alpha}$ and $\hat{\beta}$ respectively, $\text{cov}(\hat{\alpha},\ \hat{\beta})(= \text{cov}(\hat{\beta},\ \hat{\alpha}))$ is the covariance of $\hat{\alpha}$ and $\hat{\beta}$, se($\hat{\alpha}$) and se($\hat{\beta}$) are standard errors of $\hat{\alpha}$ and $\hat{\beta}$ respectively, $r_{\alpha\beta}(= r_{\beta\alpha})$ is the correlation factor between $\hat{\alpha}$ and $\hat{\beta}$.

Deviance and deviance residual

Let $\hat{\ell}$ be the maximum binomial likelihood:

$$\hat{\ell} = \prod_{j=1}^{N} \hat{p}_j^{\mu_j} (1 - \hat{p}_j)^{1-\mu_j},$$

where $\hat{p}_j (j = 1,\ 2,\ \cdots,\ N)$ is the probability given by optimal parameters $\hat{\alpha}$, $\hat{\beta}$ and γ_0

$$\hat{p}_j = \hat{p}_0(x_j) + \gamma_0 (1 - \hat{p}_0(x_j)),$$

$$\hat{p}_0(x_j) = (1 + \exp(-\hat{\alpha} - \hat{\beta} x_j))^{-1}$$

$$(j = 1, 2, \cdots, N).$$

Then we can obtain the deviance D of logistic regression:

$$D = -2 \log \hat{\ell}.$$

If we adopt the following notation:

$$d_j \equiv 2\mu_j \log \frac{1}{\hat{p}_j} + 2(1 - \mu_j) \log \frac{1}{1 - \hat{p}_j}$$

$$(j = 1, 2, \cdots, N),$$

the deviance D is given by

$$D=\sum_{j=1}^{N} d_j .$$

The deviance residual ε_j is given by

$$\varepsilon_j = \mathrm{sgn}(\mu_j - \hat{p}_j)\sqrt{d_j} \qquad (j=1, 2, \cdots, N),$$

where sgn(y) is the sign function:

$$\mathrm{sgn}(y)=\begin{cases} 1 & \text{if} \quad y>0, \\ 0 & \text{if} \quad y=0, \\ -1 & \text{if} \quad y<0. \end{cases}$$

Then we can write the deviance residual ε_j explicitly such that

$$\varepsilon_j=\begin{cases} -\sqrt{2\log\dfrac{1}{1-\hat{p}_j}} & \text{if} \quad \mu_j=0, \\ \sqrt{2\log\dfrac{1}{\hat{p}_j}} & \text{if} \quad \mu_j=1 \end{cases} \qquad (j=1, 2, \cdots, N).$$

Log-likelihood ratio test statistics of two-sample problem

Let X_1 and X_2 be two samples from the populations which have possibly different probability distributions Φ_1 and Φ_2 respectively. We shall test the following hypothesis:

null hypothesis $H_0 : \Phi_1 = \Phi_2$,

alternative hypothesis $H_1 : \Phi_1 \neq \Phi_2$.

Let $\hat{\ell}_k (k=1,\ 2)$ be the maximum binomial likelihood of samples $X_k (k=1,\ 2)$ respectively. Let X_3 be the combined sample of X_1 and X_2 :

$$X_3 = X_1 \bigcup X_2 .$$

Let $\hat{\ell}_3$ be the maximum binomial likelihood of sample X_3. Then we can define the log-likelihood ratio test statistics G such that:

$$G=-2\log\frac{\hat{\ell}(H_0)}{\hat{\ell}(H_1)}=-2\log\frac{\hat{\ell}_3}{\hat{\ell}_1\hat{\ell}_2},$$

where $\hat{\ell}(H_0)$ is the maximum binomial likelihood if H_0 is satisfied, and $\hat{\ell}(H_1)$ is the maximum binomial likelihood if H_1 is satisfied. Let D_k (k = 1, 2, 3) be the deviances which are obtained by logistic regression analysis for samples X_k (k = 1, 2, 3) respectively. D_k (k = 1, 2, 3) are given by

$$D_k = -2\log\hat{\ell}_k (k=1,\ 2,\ 3).$$

Then we have the log-likelihood ratio statistics G for the two-sample test:

$$G = D_3 - (D_1 + D_2).$$

Non-parametric bootstrap resampling

(i) Bootstrap samples X_1^{*b} and X_2^{*b}

Let X_1 be the set of binomial observations, and ε_1 be the set of deviance residuals of sample 1. Let B be the number of bootstrap samples. By adopting uniform random numbers, we draw B samples of size N_1 with replacements from ε_1 and we call them the bootstrap deviance residuals of sample 1:

$$\varepsilon_1^{*b} = \{\varepsilon_j^b \mid \varepsilon_j^b \in \varepsilon_1 (j=1,\ 2,\ \cdots,\ N_1)\} \quad (b=1,\ 2,\ \cdots,\ B).$$

Then we obtain the bootstrap sample X_1^{*b} for sample 1 such that

$$X_1^{*b} = \{x_j,\ \mu_j^b (j=1,\ 2,\ \cdots,\ N_1)\} \quad (b=1,\ 2,\ \cdots,\ B),$$

where

$$\mu_j^b = \hat{p}_0^b(x_j) + \gamma_0 (1 - \hat{p}_0^b(x_j)),$$

$$\hat{p}_0^b(x_j) = (1 + \exp(-\hat{\alpha}_1 - \hat{\beta}_1 x_j - \varepsilon_j^b))^{-1}$$

$$(j = 1, 2, \cdots, N_1 ; b = 1, 2, \cdots, B).$$

By adopting the similar method described above, we can also obtain bootstrap deviance residuals ε_2^{*b} (b = 1, 2, $\cdots$, B) and bootstrap samples X_2^{*b} (b = 1, 2, $\cdots$, B) from the set of binomial observations X_2 of sample 2.

(ii) Bootstrap sample X_3^{*b}

The bootstrap sample X_3^{*b} (b = 1, 2, $\cdots$, B) for sample 3 is obtained by the following Steps 1, 2 and 3.

Step 1:

Let ε_1^{*b} be the bootstrap deviance residuals of sample 1:

$$\varepsilon_1^{*b} = \{\varepsilon_j^b \mid \varepsilon_j^b \in \varepsilon_1 (j=1,\ 2,\ \cdots,\ N_1)\} \quad (b=1,\ 2,\ \cdots,\ B).$$

We obtain the bootstrap sample X_{31}^{*b} (b = 1, 2, $\cdots$, B) by using ε_1^{*b} (b = 1, 2, $\cdots$, B) for sample 1 and the optimal parameters $\hat{\alpha}_3$, $\hat{\beta}_3$ and γ_0 for sample 3 such that

$$X_{31}^{*b} = \{x_j,\ \mu_j^b (j=1,\ 2,\ \cdots,\ N_1)\} \quad (b=1,\ 2,\ \cdots,\ B),$$

where

$$\mu_j^b = \hat{p}_0^b(x_j) + \gamma_0 (1 - \hat{p}_0^b(x_j)),$$

$$\hat{p}_0^b(x_j) = (1 + \exp(-\hat{\alpha}_3 - \hat{\beta}_3 x_j - \varepsilon_j^b))^{-1}$$

$$(j = 1, 2, \cdots, N_1 ; b = 1, 2, \cdots, B).$$

Step 2:

By adopting the similar method in step 1, we obtain the bootstrap sample X_{32}^{*b} (b = 1, 2, $\cdots$, B) by using ε_2^{*b} (b = 1, 2, $\cdots$, B) for sample 2 and the optimal parameters $\hat{\alpha}_3$, $\hat{\beta}_3$ and γ_0 for sample 3.

Step 3:

By combining X_{31}^{*b} and X_{32}^{*b} (b = 1, 2, $\cdots$, B), we obtain the bootstrap sample X_3^{*b} (b = 1, 2, $\cdots$, B) for sample 3 such that

$$X_3^{*b} = X_{31}^{*b} \bigcup X_{32}^{*b} \quad (b = 1,\ 2,\ \cdots,\ B).$$

Hypothesis testing with the bootstrap resampling

Let D_k (k = 1, 2, 3) be the deviances obtained from the sets of binomial observations X_k (k = 1, 2, 3) respectively. Let D_k^b (k = 1, 2, 3 ; b = 1, 2, $\cdots$, B) be the bootstrap deviances obtained from the bootstrap samples X_k^{*b} (k = 1, 2, 3 ; b = 1, 2, $\cdots$, B) respectively. Let G and G^b (b = 1, 2, $\cdots$, B) be the log-likelihood ratio test statistics defined by

$$G = D_3 - (D_1 + D_2),$$

$$G^b = D_3^b - (D_1^b + D_2^b)$$

$$(b = 1, 2, \cdots, B).$$

Then we have the achieved significance level ASL:

$$ASL = \frac{\sum_{b=1}^{B} \lambda^b}{B},$$

where λ^b (b = 1, 2, $\cdots$, B) are the notations defined by

$$\lambda^b = \begin{cases} 1 & \text{if} \quad G^b \geq G, \\ 0 & \text{if} \quad G^b < G \end{cases}$$

$$(b = 1, 2, \cdots, B).$$

For avoiding ASL = 0, ASL is also defined by

$$ASL = \frac{\sum_{b=1}^{B} \lambda^b + 1}{B + 1}$$

when $\sum_{b=1}^{B} \lambda^b < \varepsilon$ (ε is a sufficiently small positive number). We can say that the null hypothesis H_0 (two samples X_1 and X_2 have common probability distributions: $\Phi_1 = \Phi_2$) is rejected if ASL is less than or equal to the significance level.

Application to psychophysical threshold estimates

(i) Mathematical notations and definitions

Let X be the set of binomial observations:

$$X = \{x_j,\ \mu_j (j = 1,\ 2,\ \cdots,\ N)\}.$$

Let Ω_i (i = 1, 2, $\cdots$, n) be the properly chosen intervals of the explanatory variable and let $\bar{x}_i$ (i = 1, 2, $\cdots$, n) be the mid-point of Ω_i. We assume that $n \leq N$. Then we define the following notations:

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad x_j \in \Omega_i, \\ 0 & \text{if} \quad x_j \notin \Omega_i \end{cases}$$

$$(i = 1, 2, \cdots, n ; j = 1, 2, \cdots, N),$$

and

$$n_i = \sum_{j=1}^{N} \delta_{ij}, \qquad m_i = \sum_{j=1}^{N} \delta_{ij} \mu_j \quad (i = 1,\ 2,\ \cdots,\ n).$$

We note that $N=\sum_{i=1}^{n} n_i$.

Let $\hat{p}(x)=p(x;\ \hat{\alpha},\ \hat{\beta},\ \gamma_0)$ be the probability given by optimal parameters $\hat{\alpha}$, $\hat{\beta}$ and γ_0 such that

$$\hat{p}(x)=\hat{p}_0(x)+\gamma_0(1-\hat{p}_0(x)),$$

$$\hat{p}_0(x)=(1+\exp(-\hat{\alpha}-\hat{\beta}x))^{-1}.$$

We define the psychophysical threshold ξ with guessing rate γ_0

$$\xi=\hat{p}^{-1}\left(\frac{1+\gamma_0}{2}\right).$$

(ii) The visual acuity test of the two-sample problem

Since we adopt the Landolt-C of four different orientations in our visual acuity test, the guessing rate γ_0 is chosen as

$$\gamma_0 = 0.25.$$

The explanatory variable x in our measurement is the logarithmic visual acuity.

Let X_1 (sample 1) and X_2 (sample 2) be samples from the populations which have possibly different probability distributions Φ_1 and Φ_2 respectively. We shall test the following hypothesis:

$$\text{null hypothesis } H_0 : \Phi_1 = \Phi_2,$$

$$\text{alternative hypothesis } H_1 : \Phi_1 \neq \Phi_2.$$

We took the data from 1 individual with no visual abnormalities in order to assess our bootstrap algorithm. The LogVA (Logarithmic Visual Acuity) is 0.3681 ± 0.0209 in complete refractive correction and we adopt this data set as sample 1. The data of sample 2 is taken in +0.50D incomplete refractive correction from the same individual of sample 1.

Table 1 and Table 2 show the observed data of sample 1 (N_1 = 120) and sample 2 (N_2 = 80) respectively. Table 3 shows sample 3 (N_3 = 200) which is constructed by the combined data of samples 1 and 2.

The logistic regression results of samples 1, 2 and 3 are shown in Table 4. Figures 1, 2, and 3 show the observed data and $\hat{p}(x)=p(x;\ \hat{\alpha},\ \hat{\beta},\ \gamma_0)$ of samples 1, 2 and 3 respectively. Psychophysical thresholds $\hat{\xi}_k$ (k = 1, 2, 3) at probability = (1 + γ_0) / 2 = 0.625 are shown in Table 4.

Now we shall prove that the samples 1 and 2 are taken from the populations which have different distributions.

Table 1 Observed data of sample 1

i	$\bar{x}_i$	n_i	m_i	m_i/n_i
1	0.156975	0	0	-
2	0.198368	20	19	0.95
3	0.244125	20	18	0.9
4	0.295278	20	18	0.9
5	0.353270	20	13	0.65
6	0.420216	20	8	0.4
7	0.499398	20	8	0.4

$N_1=\sum_{i=1}^{7} n_i = 120$

Table 2 Observed data of sample 2

i	$\bar{x}_i$	n_i	m_i	m_i/n_i
1	0.156975	13	9	0.6923
2	0.198368	14	12	0.8571
3	0.244125	13	12	0.9231
4	0.295278	13	5	0.3846
5	0.353270	13	2	0.1538
6	0.420216	14	3	0.2143
7	0.499398	0	0	-

$N_2=\sum_{i=1}^{7} n_i = 80$

Table 3 Observed data of sample 3

i	$\bar{x}_i$	n_i	m_i	m_i/n_i
1	0.156975	13	9	0.6923
2	0.198368	34	31	0.9118
3	0.244125	33	30	0.9091
4	0.295278	33	23	0.6970
5	0.353270	33	15	0.4545
6	0.420216	34	11	0.3235
7	0.499398	20	8	0.4

$N_3 = \sum_{i=1}^{7} n_i = 200$

Table 4 Logistic regression results of samples 1, 2 and 3

	sample 1 $(k=1)$	sample 2 $(k=2)$	sample 3 $(k=3)$
N_k	120	80	200
$\hat{\alpha}_k$	6.2105	4.5209	4.4349
$\hat{\beta}_k$	-16.8720	-18.3394	-14.1192
γ_0	0.25	0.25	0.25
$\mathrm{se}(\hat{\alpha}_k)$	1.4383	1.5872	0.9012
$\mathrm{se}(\hat{\beta}_k)$	4.2411	6.6365	3.0423
$\mathrm{se}(\gamma_0)$	0.0	0.0	0.0
$\hat{\xi}_k$	0.3681	0.2465	0.3141
$\mathrm{se}(\hat{\xi}_k)$	0.0209	0.0214	0.0170
D_k	115.546	90.241	224.026

$G = D_3 - (D_1 + D_2) = 18.239$

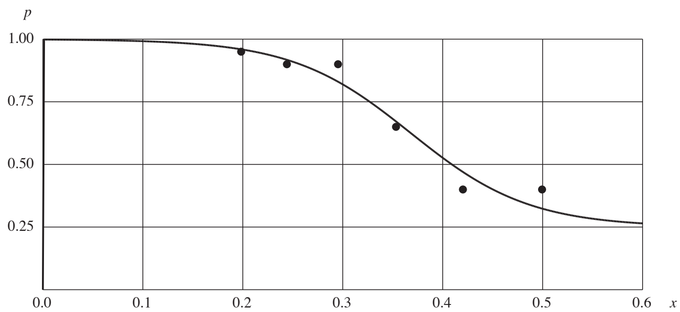

Fig. 1 Observed data and $\hat{p}(x) = p(x;\ \hat{\alpha},\ \hat{\beta},\ \gamma_0)$ of sample 1

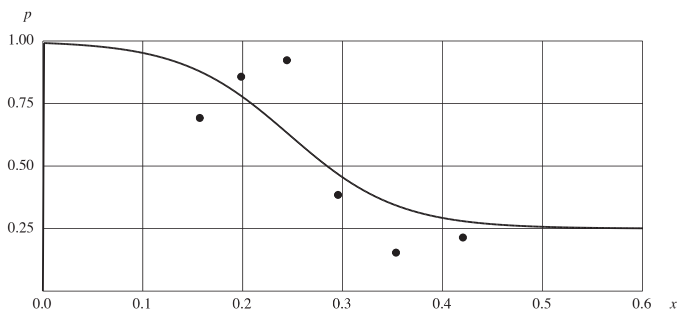

Fig. 2 Observed data and $\hat{p}(x) = p(x;\ \hat{\alpha},\ \hat{\beta},\ \gamma_0)$ of sample 2

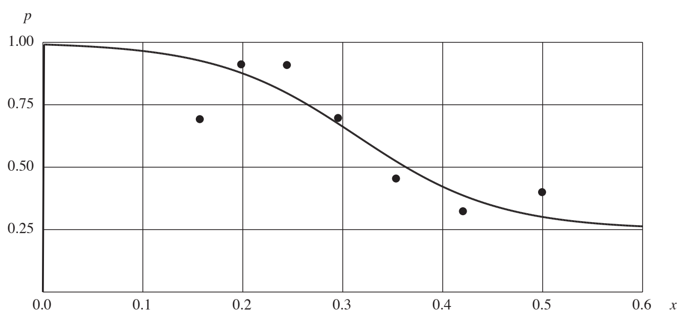

Fig. 3 Observed data and $\hat{p}(x) = p(x;\ \hat{\alpha},\ \hat{\beta},\ \gamma_0)$ of sample 3

The results of non-parametric bootstrap resampling are shown in Table 5. Since $\Lambda\left(=\sum_{b=1}^{B}\lambda_b\right)$ is small for $B = 2000$, ASL is obtained as

$$ASL = \frac{\Lambda+1}{B+1} = 0.0005.$$

This ASL shows that H_0 is rejected at a very small significant level.

Table 5 Two-sample test by bootstrap resampling

	sample 1 $(k=1)$	sample 2 $(k=2)$	sample 3 $(k=3)$
N_k	120	80	200
B_k	2000	2000	2000
mean(D_k^b)	8.962	6.612	17.862

$G^b = D_3^b - (D_1^b + D_2^b)$ $(b = 1, 2, \cdots, B)$

$\Lambda = \sum_{b=1}^{B} \lambda^b = 0$

$ASL = \frac{\Lambda+1}{B+1} = 0.0005$

(iii) The visual acuity test of one-sample problem

We use the same example described in (ii). We shall test here the parameters α, β and threshold ξ by using bootstrap resampling. Since the methods of one-sample test for α, β and ξ are the same, we show here only the case of ξ.

We adopt the following hypothesis:

$$\text{null hypothesis } H_0 : \xi = \xi_c,$$

$$\text{alternative hypothesis } H_1 : \xi \neq \xi_c,$$

where ξ_c is a prescribed value (which may be chosen from the threshold of control sample). Let $\hat{\xi}$ and se $(\hat{\xi})$ be the threshold and its standard error respectively of the (original) logistic regression. Let $\hat{z}$ be the test statistics defined by

$$\hat{z} = \frac{\hat{\xi} - \xi_c}{\mathrm{se}(\hat{\xi})}.$$

Let ξ^b ($b = 1, 2, \cdots, B$) be the thresholds obtained by the logistic regression of each bootstrap resampling. Let $\bar{\xi}$ be the mean of ξ^b ($b = 1, 2, \cdots, B$) :

$$\bar{\xi} = \frac{1}{B}\sum_{b=1}^{B}\xi^b.$$

Let se(ξ) be the standard error of ξ^b ($b = 1, 2, \cdots, B$) :

$$\mathrm{se}(\xi) = \sqrt{\frac{1}{B-2}\sum_{b=1}^{B}(\xi^b - \bar{\xi})^2}.$$

Then we have the bootstrap test statistics z^b ($b = 1, 2, \cdots, B$) as

$$z^b = \frac{\xi^b - \bar{\xi}}{\mathrm{se}(\xi)}.$$

We have the achieved significance level ASL:

$$ASL = \frac{\sum_{b=1}^{B} \lambda^b}{B},$$

where λ^b (b = 1, 2, $\cdots$, B) are the notations defined by

$$\lambda^b = \begin{cases} 1 & \text{if} \quad |\, z^b \,| \geq |\, \hat{z} \,|, \\ 0 & \text{if} \quad |\, z^b \,| < |\, \hat{z} \,| \end{cases} \qquad (b = 1, 2, \cdots, B).$$

For avoiding $ASL = 0$, ASL is also defined by

$$ASL = \frac{\sum_{b=1}^{B} \lambda^b + 1}{B+1},$$

when $\sum_{b=1}^{B} \lambda^b < \varepsilon$ (ε is a sufficiently small positive number). We can say that the null hypothesis H_0 ($\xi = \xi_c$) is rejected if ASL is less than or equal to the significance level.

In the cases of one-sample tests of α and β, we adopt the following hypothesis:

null hypothesis $H_0 : \alpha = 0$,

alternative hypothesis $H_1 : \alpha \neq 0$,

for α, and

null hypothesis $H_0 : \beta = 0$,

alternative hypothesis $H_1 : \beta \neq 0$,

for β.

One-sample tests by bootstrap resampling for α, β, ξ in samples 1, 2 are shown in Table 6.

Table 6 One-sample test by bootstrap resampling

	sample 1 $(k = 1)$	sample 2 $(k = 2)$
N_k	120	80
B_k	2000	2000
$\hat{\alpha}_k$	6.2105	4.5209
ASL_k^{*1}	0.0005	0.0050
$\hat{\beta}_k$	-16.8720	-18.3394
ASL_k^{*2}	0.0005	0.0095
$\hat{\xi}_k$	0.3681	0.2465
ASL_k^{*3}	-	0.0005
min $I_{0.95}$	0.355	0.231
max $I_{0.95}$	0.380	0.262

*1 $H_0 : \alpha_k = 0, \; H_1 : \alpha_k \neq 0 \quad (k = 1, 2)$

*2 $H_0 : \beta_k = 0, \; H_1 : \beta_k \neq 0 \quad (k = 1, 2)$

*3 $H_0 : \xi_2 = \xi_1, \; H_1 : \xi_2 \neq \xi_1$

$\Lambda_k = \sum_{b=1}^{B_k} \lambda^b \quad (k = 1, 2)$

$ASL_k = \frac{\Lambda_k + 1}{B_k + 1}$

(iv) Confidence interval of threshold

The symbols of $\hat{\xi}$, se($\hat{\xi}$), ξ^b (b = 1, 2, $\cdots$, B), $\bar{\xi}$ and se(ξ) are the same as in (iii).
Let ψ (z) be the cumulative distribution function of bootstrap resampling defined by

$$\psi(z) = \frac{1}{B}\sum_{b=1}^{B}\varphi^b(z) \quad (-\infty < z < +\infty),$$

where φ^b (z) (b = 1, 2, $\cdots$, B) are functions of z :

$$\varphi^b(z) = \begin{cases} 1 & \text{if} \quad z \geq z^b, \\ 0 & \text{if} \quad z < z^b \end{cases}$$

$$(b = 1, 2, \cdots, B).$$

We note that ψ (z) satisfies

$$\psi(z) \to 0 \quad (z \to -\infty),$$

$$\psi(z) \to 1 \quad (z \to +\infty).$$

Then we can obtain the confidence interval I_ρ of confidence coefficient ρ ($0 < \rho < 1$) such that

$$I_\rho:\ \hat{\xi} - \psi^{-1}\left(\frac{1-\rho}{2}\right) \cdot \text{se}(\xi) \leq \xi \leq \hat{\xi} + \psi^{-1}\left(\frac{1+\rho}{2}\right) \cdot \text{se}(\xi).$$

The confidence intervals $I_{0.95}$ of threshold ξ for samples 1 and 2 are shown in Table 6.

Concluding remarks

We proposed the bootstrap resampling algorithm for the psychophysical threshold estimates. Main properties of our algorithm are summarized in the following:
(i) the logistic regression including the guessing rate,
(ii) the non-parametric bootstrap resampling with log-likelihood ratio statistics for two-sample testing,
(iii) the non-parametric bootstrap resampling for one-sample testing to certify the values of parameters and threshold obtained by logistic regression.

We applied our bootstrap algorithm to the visual acuity test problem. Our algorithm does not require the identity of the number of observations between two samples. We can say that the bootstrap resampling provides a useful tool which has the flexibility of sampling in actual visual acuity measurements.

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